Estimation Noise in Portfolio Optimization with Absolute Deviation

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Abstract

Portfolio selection has a central role in finance theory and practical applications. The classical approach uses the standard deviation (variance) as risk measure, while several other alternatives have also been introduced in the literature. Due to its computational advantages, portfolio optimization based on absolute deviation looks particularly interesting and it is widely used in practice. For the practical implementation of any variant, however, one needs to estimate the parameters from finite return series, which inevitably introduces measurement noise that in turn affects the portfolio selection. Although much research has concentrated on investigating the noise in the classical model, little attention has been devoted to the case of absolute deviation. In this paper, we study the effect of estimation noise in the case of absolute deviation based portfolio optimization. We show that the key parameter determining the effect of noise is the ratio of the time series length and portfolio size and that the level of noise is higher than in the classical variance-based model. This points out the importance of finding out when theoretically ,,better" portfolio selection models can truly outperform the basic ones.

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1 Introduction

Since the seminal work of Markowitz (1952, 1959) portfolio selection has gained a central role in finance, both in theory and practical applications (see e.g. Elton and Gruber (1995) and the numerous references therein). Mean–variance portfolio selection along with the subsequently developed Capital Asset Pricing Model (CAPM) form the pillars of modern investment theory and have led to important investment and risk management applications such as, for example, capital allocation or risk adjusted performance measurement.

However, it has been clear since the very outset of the theory that its practical implementation and use is not so simple and straightforward. First, the input parameters in the optimization problem (expected returns of the assets and the covariance matrix of returns) are not given, but have to be estimated. Very usually, for determining ,,expected" returns in practice, one has to resort to analyst's forecasts based on different quantitative and non-quantitative information, and often on methodologies built on shaky or at least subjective elements. A portfolio selection framework that takes into account the nature of these ,,expected return" figures by considering them ,,subjective views" rather than estimates has been introduced by Black and Litterman (1992). On the other hand, in several applications, one attempts to minimize risk without any reference to expected returns (e.g. in several hedging problems or benchmark tracking); in these cases expected return estimates are not needed and consequently the above mentioned problems do not arise at all.

In any case though, covariance matrices of financial returns do need to be estimated, which is generally done from financial time series data. Since one has to estimate $O(N^2)$ covariance matrix elements (N denotes the number of assets) from NT datapoints (T denotes the length of the time series), it is clear that unless $T \gg N$ (which is usually not the case in practical applications) these estimates will contain considerable noise, which can in turn adversely affect the determination of the optimal portfolio. This has been recognized very early in the literature and several procedures have been introduced in order to reduce the estimation error, for example factor models or Bayesian shrinkage estimators. By lowering the number of free parameters to be estimated (the dimensionality of the problem) most of these techniques can achieve a significant reduction of noise.

Second, since mean–variance portfolio selection requires the minimization of a quadratic function (in the asset portfolio weights) subject to different constraints, it usually leads[1] to a convex optimization problem[2] that needs to be solved numerically. Although (due to significant advances in computing technology since the times when mean–variance portfolio selection started to be applied in practice) this does not constitute anymore an impassable barrier to its practical implementation, for large portfolios

[1] Except the simple case when the only constraints are the budget constraint and the one on the expected returns, when the problem can be solved analytically.

[2] For example, if in addition to the above two constraints, one imposes the simple requirement of non-negative portfolio weights (short-selling is prohibited).

it can still require considerable resources and non-standard optimization techniques. It can be therefore interesting to consider alternatives to the classical mean–variance optimization, most interestingly such that the idea of mean–risk optimization is preserved (but instead of the standard deviation with other measures for risk). Most importantly, like the original mean–variance framework (with suitable risk measures) this remains compatible (under certain conditions) with expected utility maximization.

One such a portfolio optimization framework has been advanced by Konno and Yamazaki (1991) and it is based on absolute deviation as a risk measure. The risk of a portfolio (of weights w_i) of N assets with returns described by random variables r_i of means μ_i is given in this case by

$$\mathrm{E}\Big(\Big|\sum_i w_i(r_i-\mu_i)\Big|\Big), \tag{1}$$

where $\mathrm{E}(\cdot)$ denotes the expected value of the random variable in the argument and $i = 1, 2, \ldots, N$. In practice (like in the mean–variance case) one has to minimize (subject to different constraints) an estimate of this based on a sample of finite time series; for this

$$\frac{1}{T}\sum_t \Big|\sum_i w_i(r_{it}-\mu_i)\Big| \tag{2}$$

is most usually used, where r_{it} denotes the return on asset i at time t ($t = 1, 2, \ldots, T$) and $\mu_i = \frac{1}{T}\sum_t r_{it}$. The main advantage of the above mean–absolute deviation portfolio optimization model over the classical mean–variance optimization is computational: in the case of mean–absolute deviation the problem reduces to a linear programming problem (even if in addition to the usual budget constraint and the constraint on expected returns one introduces several other (linear) constraints such as for example restriction on short-selling or different other more complex limits), and solving this linear programming problem requires significantly less computational burden and only standard optimization techniques compared to the classical mean–variance optimization (which reduces to quadratic optimization).

It seems that (most probably due to its computational advantage) mean–absolute deviation portfolio selection has gained important ground also in practice. For example, Algorithmics, a leader in risk management solutions, has built its portfolio optimization tool on absolute deviation as risk measure[3] (Dembo and Rosen 2000, Algorithmics 2002). Besides the usual mean–risk optimization, the software can be used to minimize risk without constraints on expected returns, providing solutions for example for benchmark tracking, portfolio compression or different hedging or pricing problems. With this portfolio optimization based on absolute deviation has become widely used in practice.

However, little attention has been given to the estimation error in this case. Even if we consider only the situation in which expected returns drop out of the problem,

[3]More precisely, the user can choose from different forms of absolute deviation (e.g. considering all or only the negative returns, respectively) or a form of ,,maximal loss" (which also leads to a linear programming problem).

there might be considerable noise originating from the finiteness of the time series in Eq. (2). When the variance is used for measure of risk, the noise enters the problem through the covariance matrix estimated from the (finite) time series of returns; in the case of absolute deviation (although it might not be obvious at first sight) similar noise must arise from the finiteness of the time series (through Eq. (2)). Since (as briefly discussed above in the case of the variance as measure of risk) noise can significantly affect the choice of the optimal portfolio, it would be therefore important to know the magnitude of the effect.

Since very early, much research has focused on the estimation noise and the performance of different noise reduction techniques in the case of mean–variance optimization (e.g. Elton and Gruber 1973, Eun and Resnick 1984, Chan, Karceski and Lakonishok 1999). In contrast with the empirical approach used in the literature, we proposed in an earlier paper (Pafka and Kondor 2002) a model (simulation) based approach and introduced an appropriate metric for the effect of noise, and we used this framework to systematically investigate the effect of noise in the problem of variance-based risk minimization. We showed that the effect of noise depends essentially on the ratio T/N of the length of the time series and the size of the portfolio (Pafka and Kondor 2003), and that indeed, dimension reduction techniques can be very efficient in reducing this estimation noise (Pafka and Kondor 2004). However, little work has been done in investigating the effect of noise in the mean–absolute deviation portfolio optimization problem (except Simaan 1997).

In this paper we extend our earlier methodology to investigate the effect of noise in the risk minimization problem based on absolute deviation. Very much like in Simaan (1997), we find that the noise (originating from the finiteness of the return series used in the optimization) can amount to significant levels and that this level is higher than in the case of variance-based risk optimization. In addition, however, we manage to identify the key factors determining the level of noise in the mean–absolute deviation portfolio optimization and show that (for a large choice of the random process generating the time series) the effect of noise depends essentially only on the ratio T/N.

2 Results and Discussion

For our investigations we use the following adaptation of the simplified portfolio optimization framework advanced by Pafka and Kondor (2003): the portfolio risk (estimate) $\frac{1}{T}\sum_t |\sum_i w_i r_{it}|$ is minimized under the budget constraint $\sum_i w_i = 1$, where r_{it} represents (normally distributed) surrogate return series generated using different covariance structures. The ,,optimal" portfolio in the presence of noise is determined by solving the above minimization problem (which reduces to linear programming), while the ,,true" optimal portfolio is determined by solving the minimization of $\mathrm{E}(|\sum_i w_i r_i|)$ under the same budget constraint, which as shown in Konno and Yamazaki (1991) is equivalent to solving the corresponding variance minimization problem (if returns are normally distributed). Using the metric introduced in Pafka and Kondor (2002), we

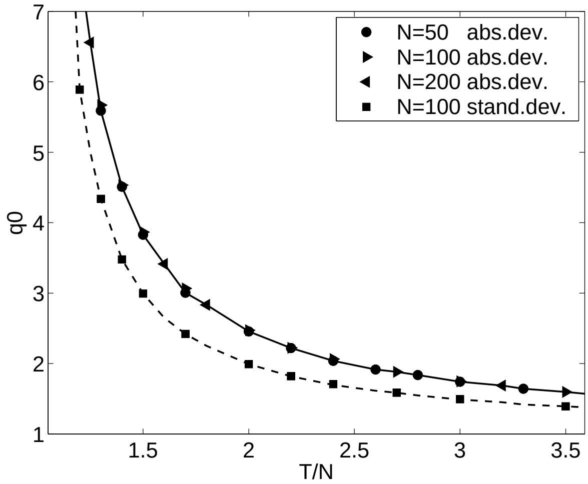

FIG. 1: q_0 as a function of T/N for different N, T and different covariance structures.

quantify the effect of noise by q_0, the ratio of the risk (in this case the absolute deviation) of the optimal portfolio in the presence of noise and the risk of the true optimal portfolio[4].

For different values for the portfolio size N and time series length T, and for different covariance structures $\sigma_{ij}^{(0)}$ (Pafka and Kondor 2004), we determined the effect of noise (q_0) using Monte Carlo simulations. The results are summarized in Fig. 1. It can be seen from the figure that for large sizes, the effect of noise depends essentially only of T/N (for a large choice of the covariance structure of returns). Pafka and Kondor (2003) found a similar dependence in the classical variance-based case[5], also shown in the figure. We repeated our measurements of q_0 for different non-Gaussian distributions of returns (but with finite second moments) and obtained similar results, namely scaling in T/N on the same curve. It can be therefore concluded that the key factor determining the effect of noise in the absolute deviation based portfolio optimization (similarly to the classical variance-based case) is T/N. Moreover, (in both cases) as T approaches N from above, q_0 increases and diverges, anticipating the fact that for $T < N$ the optimization problem becomes degenerate (and meaningless from a practical point of view).

The other striking feature of the results presented in Fig. 1 is that for the same choice of the input parameters the level of noise in the absolute deviation based optimization is higher than in the classical case with standard deviation. An interesting

[4]We emphasize that by risk (of a portfolio of weights w_i) we mean $\mathrm{E}\left(\left|\sum_i w_i r_i\right|\right)$, which in the case of normally distributed returns is proportional to the standard deviation of $\sum_i w_i r_i$, i.e. to $\left(\sum_{ij} w_i \sigma_{ij}^{(0)} w_j\right)^{1/2}$, where $\sigma_{ij}^{(0)}$ is the covariance matrix used for generating the return series.

[5]In addition to simulation results, using statistical tools from theoretical physics (random matrix theory) it is possible to derive a closed, analytical formula for q_0 in the $N \to \infty$ limit: $q_0 = 1/\sqrt{1 - N/T}$ (Pafka and Kondor 2003), which matches well the simulation results already for $N \sim 50$.

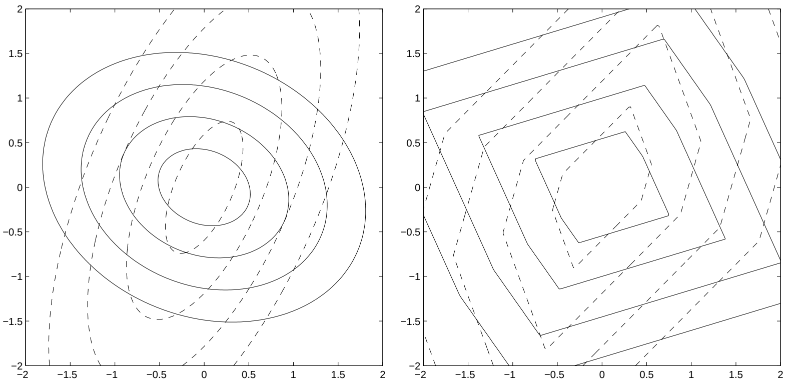

FIG. 2: Level surfaces of standard (*left*) and absolute deviation (*right*).

visual (although not rigorous) explanation for this can be obtained by analyzing the form of the iso-risk level surfaces (in the space of portfolio weights) of standard and absolute deviation, respectively. If returns are, for example, independent normally distributed, the ,,true" iso-risk surfaces of both standard and absolute deviation are concentric spheres. However, when risk must be estimated from finite return series, the iso-risk surfaces become ,,deformed" and, in general, higher level of noise will cause more significant deformation. Typical level surfaces (contour lines in the case of two-dimensional portfolios) of standard and absolute deviation are shown in Fig. 2 for $N = 2$ and $T = 4$. In can be seen from the figure that in both cases the deformation (relative to the ideal case of concentric circles) can be significant. On the other hand, in the case of standard deviation the iso-risk surfaces remain ,,round" (ellipses) while in the case of absolute deviation they become ,,edgy" (polygons). Since the solution to the considered risk minimization problems is at the intersection of these level surfaces and the linear budget constraint, it is clear that in the case of absolute deviation the solution is less ,,stable," since a small change in the orientation of a polygon (polyhedron in general) can make the solution make a larger move on the budget constraint than a similar change in the orientation of an ellipse (ellipsoid). It is clear now that ,,linearizing" the problem (which reduces the problem to linear programming, but makes the iso-risk levels edgy polyhedrons) comes not only with the computational advantage of linear programming, but also with the inevitable increase in instability and estimation noise.

3 Conclusion

Due to its computational advantages, portfolio optimization based on absolute deviation as risk measure (instead of the standard deviation of the classical approach) seems to have recently gained ground in practice. Although much research has focused on the effect of estimation noise in the classical (standard deviation based) problem, little concern has been attributed to the case of absolute deviation. In this paper we analyzed the effect of estimation noise in the absolute deviation based portfolio optimization.

We found that the level of noise can be significant and it is in general higher than in the case of standard deviation based optimization. This points out a possible trade off between computational advantage and noise level, which should be carefully analyzed whatever risk measure one intends to use in practice.

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