Understanding interarrival and interdeparture time statistics from interactions in queuing systems

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Abstract

We discuss the statistics of long queues, in which the interdeparture time statistics is dominated by spatial interactions among the elements in a queue rather than the arrival or exit processes. Based on a Fokker–Planck approach, it is possible to calculate the stationary distance distribution among the elements in a queue as a function of their interaction potential. The results relate to the ones known from Random Matrix Theory. Together with the velocity distribution, one can determine the time-gap distribution as well. This yields an analytical approach to the interdeparture and interarrival time distributions of queuing systems with spatially interacting elements. While these distributions are usually determined from empirical data or from theoretical assumptions about the arrival or exit process, we offer here an alternative interpretation of interdeparture time distributions as an effect of interactions in a queue. This is relevant for the understanding of traffic and production systems and for the optimization of the statistical behavior of some queuing systems.

Keywords: Queuing theory; One-dimensional gas; Driven many-particle system; Fokker–Planck equation; Random matrix theory; Time-gap distributions; Interdeparture time distributions

1. Introduction

Material flows are one of the main subjects of physics. The dynamics of particle flows has been described on a fluid-dynamic level \cite{1,2} by gas-kinetic approaches \cite{1,3}, and by microscopic models \cite{1,4}, e.g., molecular dynamic approaches. Similar methods are recently applied to gain a better understanding of supply chains \cite{5–11}. While the classical, queuing theoretical approach \cite{8} has mainly delivered results such as the expected waiting time for a stationary queue, the dynamics of interacting units in a supply chain is attracting more and more attention \cite{9–11}. A fluid-dynamic approach to supply networks has, for example, yielded a new interpretation of business cycles based on the bull-whip effect, according to which variations of the inventory are increasing from one supplier to the next one in the line \cite{5,12–14}.

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Interestingly, the fluid-dynamic supply chain model is closely related to “microscopic” follow-the-leader models of vehicle traffic. These equations will be the focus of this paper. We will present formulas for the stationary velocity and distance distributions of vehicles, or more generally speaking, of interacting-driven particles. This approach yields a generalization of queuing theory to the case of interacting elements in the queue and, thereby, offers an alternative interpretation of interarrival and interdeparture time distributions with potential applications to the optimization of production systems.

Our paper is organized as follows: Section 2 presents a driven many-particle model with particle interactions in one dimension. It also gives a Fokker–Planck formulation which allows one to derive the stationary velocity and distance distributions. Section 3 then shows how these results can, for example, be used to derive the interaction potential of vehicles on a freeway or to understand the gap distributions in some production or other queuing systems. Section 4 discusses, how this approach relates to classical queuing theory, in which the interarrival or interdeparture time distributions are a result of the stochastic arrival and exit processes. Finally, Section 5 summarizes our results and presents an outlook.

2. Model of driven interacting particles in one dimension

Let us assume a system of \( n \) point-like particles \( i \) with positions \( x_i(t) \) and actual velocities \( v_i(t) \) at time \( t \). Their motion is naturally described by the equation

\[
\frac{dx_i}{dt} = v_i(t),
\]

while their velocity change \( dv_i/dt \) will be assumed according to the following acceleration equation:

\[
\frac{dv_i}{dt} = \frac{v_0 - v_i}{\tau} + f(s_i) - \gamma f(s_{i-1}) + \xi(t). \tag{2}
\]

Herein, \( (v_0 - v_i)/\tau \) is usually called the driving term. \( v_0 \) means the maximum velocity of the particles, \( \tau \) an acceleration or relaxation time, and \( f(s_i) \) the repulsive interaction force as a function of the distance \( s_i(t) = x_{i+1}(t) - x_i(t) \). \( \gamma = 1 \) corresponds to the case of symmetrical interactions fulfilling Newton’s law of “actio = reactio”, while \( \gamma = 0 \) corresponds to forwardly directed interactions as for vehicle traffic (where the cars play the role of the particles). \( \xi(t) \) denotes a Gaussian white noise, i.e., we have the relationships

\[
\langle \xi(t) \rangle = 0 \quad \text{and} \quad \langle \xi(t)\xi(t') \rangle = D\delta(t-t'). \tag{3}
\]

The diffusion coefficient \( D \) determines the strength of the fluctuations \( \xi_i(t) \).

2.1. The Fokker–Planck equation and its stationary solution

In order to determine the velocity and distance distribution functions of the driven many-particle system, it is reasonable to reformulate the stochastic process (2) in terms of a Fokker–Planck equation. It reads

\[
\frac{\partial P}{\partial t} = \sum_{i=1}^{n} \left\{ -\frac{\partial}{\partial s_i} \left[ (v_{i+1} - v_i) P \right] - \frac{\partial}{\partial v_i} \left[ \frac{W(s_i, s_{i-1})}{\tau} P \right] + D \frac{\partial^2 P}{\partial v_i^2} \right\}, \tag{4}
\]

where we have used the abbreviation

\[
W(s_i, s_{i-1}) = v_0 + \tau[f(s_i) - \gamma f(s_{i-1})]. \tag{5}
\]

In addition, we will define the distance-dependent interaction potential \( U(s_i) \) via

\[
f(s_i) = -2 \frac{dU(s_i)}{dx_i} = \frac{2}{1 + \gamma} \frac{dU(s_i)}{ds_i}, \tag{6}
\]

i.e.,

\[
U(s_i) = \frac{1 + \gamma}{2} \int_0^{s_i} ds' f(s'). \tag{7}
\]
In the following, we will assume that the distances and velocities are mutually uncorrelated, i.e., \(\langle s_i(t)v_j(t)\rangle = 0\) for all \(i\) and \(j\). By means of extensive calculations, it can then be shown [15] that the above Fokker–Planck equation has the stationary solution
\[
P(s_1, \ldots, s_n, v_1, \ldots, v_n) = \prod_{i=1}^{n} [Q(s_i)R(v_i)]
\]  
with the distance distribution (density function)
\[
Q(s_i) = Ae^{-\frac{U(s_i)}{\theta} - Bs_i}
\]
and the velocity distribution (density function)
\[
R(v_i) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{(v_i - V)^2}{2\theta}},
\]
if the fluctuation–dissipation theorem
\[
\frac{1}{\theta} = \frac{2}{D\tau}
\]
is fulfilled. Herein,
\[
\theta = \int_{-\infty}^{\infty} dv(v - V)^2 R(v)
\]
is the velocity variance of the particles, and
\[
V = \int_{-\infty}^{\infty} dv v R(v)
\]
their average velocity. The parameter \(A\) in Eq. (9) is a normalization constant obtained via the constraint
\[
\int_{0}^{\infty} ds_i Q(s_i) = 1.
\]
The Lagrange parameter \(B\) determines the particle density \(\rho\). It is given by
\[
\int_{0}^{\infty} ds_i s_i Q(s_i) = \frac{1}{\rho}.
\]
Rewriting these equations, we get
\[
\frac{1}{A} = \int_{0}^{\infty} ds_i e^{-\frac{U(s_i)}{\theta} - Bs_i}
\]
and
\[
\int_{0}^{\infty} ds_i s_i e^{-\frac{U(s_i)}{\theta} - Bs_i} = \frac{1}{A\rho} = \frac{1}{\rho} \int_{0}^{\infty} ds_i e^{-\frac{U(s_i)}{\theta} - Bs_i}.
\]
For a given potential \(U(s)\), relaxation time \(\tau\), and noise parameter \(D\) (or a given variance \(\theta = \frac{1}{2}D\tau\)), \(B\) can be determined via the condition
\[
\frac{\int_{0}^{\infty} ds_i s_i e^{-\frac{U(s_i)}{\theta} - Bs_i}}{\int_{0}^{\infty} ds_i e^{-\frac{U(s_i)}{\theta} - Bs_i}} = \frac{1}{\rho},
\]
and the normalization constant is then calculated via Eq. (16).

More interesting, however, is the inverse problem of determining the interaction potential and possibly the parameters \(D\) and \(\tau\) from the observed distribution of particles. This will be applied to several multi-particle systems in the next section.
3. Relevance for traffic and production systems

3.1. Transforming distance into time-gap distributions and vice versa

The considerations of the previous section have shown that, for determining the interaction potential, the distribution \( Q(s) \) of spatial distances \( s \) is a crucial quantity. Frequently, however, one gets empirical data for the related time-gap distributions \( S(\tau) \). For instance, for vehicular traffic, the time-gap can be obtained from the difference of passage times of stationary detectors, if single-vehicle data are available.

It is therefore necessary to calculate approximatively the distance distribution \( Q(s) \) from a given time-gap distribution \( S(\tau) \). Knowing the velocity distribution \( R(v) \), and assuming the velocities and distances to be mutually independent as in Eq. (8), it is possible to approximatively determine the distance distribution \( Q(s) \) belonging to a given time-gap distribution \( S(\tau) \) as follows.

Let \( R(v)Q(s)dv\,ds \) be the probability of finding a particle with a velocity between \( v \) and \( v + dv \) with a distance between \( s \) and \( s + ds \) to the preceding particle. The probability \( Q(s)\,ds \) of finding a distance between \( s \) and \( s + ds \) is then obtained by integration over all time gaps \( \tau \) and velocities \( v \) which belong to a given distance \( s = \tau v \). Correspondingly, we get

\[
Q(s) = \int_{-\infty}^{\infty} dv \int_{0}^{\infty} d\tau R(v)S(\tau)\delta(s - v\tau)
\]
\[
= \int_{-\infty}^{\infty} dv R(v) \int_{0}^{\infty} d\tau S(\tau) \frac{1}{v} \delta(\tau - s/v)
\]
\[
= \int_{-\infty}^{\infty} dv R(v) S(s/v) \frac{1}{v}.
\]

(19)

Now let us express the velocity by its mean \( V \) and deviation \( w = v - V \), expand the time-gap distribution to second order around \( s/V \), and evaluate the integrals obtained by inserting the velocity distribution \( R(v) \). In this way, we get

\[
Q(s) = \int_{-\infty}^{\infty} dw R(V + w)S(s/(V + w)) \frac{1}{V + w}
\]
\[
\approx \int_{-\infty}^{\infty} dw \frac{1}{\sqrt{2\pi}0} e^{-w^2/(20)} \left\{ \frac{1}{V} S\left(\frac{s}{V}\right) - w \left[ \frac{1}{V^2} S\left(\frac{s}{V}\right) + \frac{s}{V^3} S'\left(\frac{s}{V}\right) \right] \right.
\]
\[
+ \frac{w^2}{2} \left[ \frac{2}{V^3} S\left(\frac{s}{V}\right) + \frac{4s}{V^4} S'\left(\frac{s}{V}\right) + \frac{s^2}{V^5} S''\left(\frac{s}{V}\right) \right] \right\}
\]
\[
= \frac{1}{V} \left\{ S\left(\frac{s}{V}\right) + \frac{\theta}{V^2} \left[ S\left(\frac{s}{V}\right) + \frac{2s}{V} S'\left(\frac{s}{V}\right) + \frac{s^2}{2V^2} S''\left(\frac{s}{V}\right) \right] \right\}.
\]

(20)

According to this, the distance distribution can be expressed as an expansion with respect to the square of the velocity variation coefficient \( \theta/V^2 \). To lowest order, it is given by \( Q(s) = 1/V S(s/V) \).

Conversely, when the velocity distribution \( R(v) \) is known, it is possible to determine the time-gap distance distribution \( S(\tau) \) belonging to a given distance distribution \( Q(s) \):

\[
S(\tau) = \int_{-\infty}^{\infty} dv \int_{0}^{\infty} ds Q(s)R(v)\delta(\tau - s/v)
\]
\[
= \int_{-\infty}^{\infty} dv R(v)Q(\tau v)v
\]
\[
= \int_{-\infty}^{\infty} dw R(V + w)Q((V + w)\tau)(V + w)
\]
\[
\approx V \left\{ Q(V\tau) + \frac{\theta}{V^2} \left[ V\tau Q'(V\tau) + \frac{V^2\tau}{2} Q''(V\tau) \right] \right\}.
\]

(21)
3.2. Determination of the interaction potential of vehicles

For $\gamma = 0$, the coupled stochastic differential equations (2) have been suggested for the description of the interactive dynamics of vehicles on freeways. Therefore, the stationary solution (8) of the related Fokker–Planck equation has been used to determine the interaction potential of vehicles [16]. Previous investigations, however, were very restricted in the choice of the interaction potential, as the parameters $A$ and $B$ could be mostly not analytically calculated.

In the following, we will propose a simple and generally applicable numerical procedure for determining the interaction potential from single-vehicle data of homogeneous traffic, without restriction of its functional form.

Combining Eqs. (6), (8), and (9), and setting $\gamma = 0$, we get the following relation between the single-particle interaction force and the observed distance distribution $Q(s_i)$:

$$f(s_i) = 2\frac{dU(s_i)}{ds_i} = -2\theta \left( \frac{d\ln Q(s_i)}{ds_i} + B \right).$$

This expression, which serves as a basis for solving the “inverse scattering” problem, contains the measurable velocity variance $\theta$ and the unknown Lagrange parameter $B$. While $\theta$ does not depend on the traffic density (if $D$ is not dependent on $v$ or $r$), the parameter $B$ increases with the density and has the asymptotics $B \approx r$ for $r(1/r) \ll v_0$, i.e., for small densities $r$, where interactions are negligible (cf. Eq. (26) below).

For intermediate densities around $r = 30$ vehicles/km/lane, the interactions generally play an essential role. However, there are still some vehicles with large gaps to their predecessors, that de facto do not experience significant interactions, resulting in an exponential tail of the density function of the distance distribution. This can be used to determine the coefficient $B$ via

$$B \approx -\frac{d\ln Q(s_i)}{ds_i} \text{ for } s > s_c,$$

where the crossover distance $s_c$ (of the order of 100 m) has to be determined from the data.

Eqs. (22) and (23) allow one to empirically determine the interaction force $f(s)$ from the observed distance distribution $Q(s)$ and the velocity variance $\theta$ of homogeneous traffic of density $r$. This requires that (i) the distance distribution is significantly different from that of non-interacting particles (cf. (26) below), and (ii) it has an exponential tail. The first condition restricts the density range for the observed data from below, and the second from above. This results in a useful density range $\rho_{c1} < r < \rho_{c2}$ for the determination of the interaction force $f(s)$, where $\rho_{c1}$ is of the order of 5 vehicles/km/lane and $\rho_{c2} \approx 25$ vehicles/km/lane. With the knowledge of the interaction force $f(s)$, the interaction potential can be obtained by integration (see Eq. (7))

$$U(s) = \frac{1 + \gamma}{2} \int_0^s ds' f(s').$$

In Fig. 5, we have applied the above method to single-vehicle data from a section of the Dutch freeway A9 (Fig. 1). In order to evaluate stationary and homogeneous traffic situations only (disregarding non-equilibrium situations such as stop-and-go waves), we preprocess the data as follows. First, we select
consecutive data blocks of \( N = 50 \) vehicles on the left lane of all detector cross sections. Then, we remove car–truck, truck–car, and truck–truck sequences and calculate, for the remaining car–car pairs of each block, the average velocity, velocity variance, and density (via the hydrodynamic relation \( \rho = Q/V \)). Moreover, we only take into account data blocks for which the difference in the velocity variance is below \( \Delta \theta = 15 \text{m/s}^2 \). Finally, we subdivide the remaining data blocks belonging to uncongested traffic into three density classes given in Table 1.

Figs. 2 and 3 show the resulting velocity and distance distributions for the three density classes based on Table 1, and Fig. 4 displays the interaction force calculated from Eqs. (22) and (23). Our main results

<table>
<thead>
<tr>
<th>Class</th>
<th>( \rho ) (1/km)</th>
<th>( \theta ) (m(^2)/s(^2))</th>
<th>( B ) (1/km)</th>
<th>( f ) (m/s(^2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4–10</td>
<td>8.5</td>
<td>0.0073</td>
<td>−0.032</td>
</tr>
<tr>
<td>2</td>
<td>10–20</td>
<td>8.7</td>
<td>0.0103</td>
<td>−0.051</td>
</tr>
<tr>
<td>3</td>
<td>20–30</td>
<td>10.1</td>
<td>0.0177</td>
<td>−0.091</td>
</tr>
</tbody>
</table>

Fig. 2. Velocity distributions \( R(v) \) for the three density intervals given in Table 1 in (a) linear and (b) semi-logarithmic representation. The mean speeds are \( v_e = 31.8, 30.1, \) and 26.403 m/s, respectively.

Fig. 3. Distance distributions \( Q(s) \) for the density classes given in Table 1.
are as follows:

- For all density classes, the velocity distribution is nearly Gaussian, in accordance with Eq. (10). Moreover, the velocity variances are almost identical (cf. Table 1), which is consistent with density-independent values of $\tau$ and $D$ (the situation is to be expected to be different for high densities, though).
- The interaction potential calculated independently for each of the three density classes turns out to look similar, in agreement with our theory (see Eq. (2) and Fig. 3(b)).
- The calculated values for the interaction force corresponds to realistic accelerations of the order of $1 \text{m/s}^2$.

3.3. Comparison with some results from classical queuing theory

In most queuing theoretical studies, the statistics of the time gaps between successive “particles” is treated as a (often undesired) fact, which is built into corresponding models by means of assumed interarrival or interdeparture time distributions. These distributions are either specified according to theoretical assumptions or empirically calibrated, but in most cases no satisfactory explanation is given for them. We suggest that, at least in some cases, the interarrival or interdeparture time distributions can be understood as a result of non-linear particle interactions. This approach is quite natural for the interpretation of the time-gap or distance distribution of successive vehicles on a street. For production processes, the presence of direct interaction forces is not always obvious. However, the interaction forces may also be of indirect nature, e.g. mediated by the surroundings such as in the case of a worker who is busy dealing with one product, while another one has to wait. Many similar situations may be thought of, in which interactions between two successively generated products occur, which play the role of the past orders in our theoretical treatment.

In contrast to classical queuing theory, which assumes no characteristic interactions among the waiting elements, but certain interarrival and interdeparture time distributions, we suggest that the statistics, at least in some cases, may be understood within an extended queuing theory considering particle interactions. For illustrative purposes, let us now discuss some special cases assuming small velocity variances $\theta \neq 0$ satisfying $\theta \ll V^2$, which implies the simplified relationships

$$S(\tau) = VQ(V\tau) \quad \text{and} \quad Q(s) = \frac{1}{V}S\left(\frac{s}{V}\right).$$  (25)
(1) In the case of no interaction among particles, i.e. \( U(s) = 0 \), we just find an exponential distance distribution
\[
Q(s) = A e^{-B s}
\]
and the exponential time-gap distribution
\[
S(\tau) = V A e^{-B V \tau},
\]
where \( B = \rho \) and \( A = \rho \).
Note that exponential interdeparture time distributions are characteristic for Poisson processes. These are found if many stochastic events occur independently from each other at a certain rate. Without an interaction potential (i.e. enough lanes to overtake whenever desired), we do not have any interdependence between arriving vehicles, which can be imagined to start from a far distance at a certain rate, subject to independent fluctuations. This is, why we expect a Poissonian statistics of arriving vehicles under such conditions.

(2) For a logarithmic potential \( U(s) = -(k - 1) \ln(s/s_0) = -\ln(s/s_0)^{k-1} \), where \( k \) is an exponent and \( s_0 \) some unit length, we find the distance distribution (density function)
\[
Q(s) = A s(s_0)^{k-1} e^{-B s}
\]
and the time-gap distribution (density function)
\[
S(\tau) = A V(s_0)^{k-1} \tau^{k-1} e^{-\lambda \tau},
\]
where \( \lambda = BV \), \( B = k \rho \), and
\[
A = \frac{\lambda^k (s_0)^{k-1}}{(k - 1)!} \frac{B V}{V^k} = \frac{B^k (s_0)^{k-1}}{(k - 1)!} V^k.
\]
For \( k > 1 \), the expression for \( S(\tau) \) can be obviously identified with another fundamental time-gap distribution of queuing theory, namely the Erlang distribution. In contrast, for values \( k < 1 \) we obtain a power-law distribution with an exponential cutoff as in an experimental study and priority queuing model by Barabási [17], and even in the correspondence patterns of Darwin and Einstein [18].
Note that the formula (29) for the Erlang distribution can be easily generalized to the Gamma distribution with non-integer values of \( k \), essentially by replacing \( (k - 1)! \) in Eq. (30) by \( \Gamma(k) \).

(3) In the case of a quadratic interaction potential \( U(s)/\theta = c(s - s_0)^2 \), where \( s_0 \) denotes a preferred distance, we get
\[
Q(s) = A e^{-c(s-s_0)^2 - B s}
\]
and
\[
S(\tau) = A V e^{-c(V\tau-s_0)^2 - B V \tau}.
\]
Here, the parameters are determined by
\[
B = 2c \left( s_0 - \frac{1}{\rho} \right) \quad \text{and} \quad A = \sqrt{\frac{c}{\pi}} e^{-c[1/\rho^2 -(s_0)^2]},
\]
if \( c \) satisfies \( c \gg \rho^2 / 2 \), i.e. the probability of negative values of \( s \) is negligible. Remarkably, the resulting gap distribution
\[
Q(s) = \sqrt{\frac{c}{\pi}} e^{-c(s-1/\rho)^2}
\]
does not depend on \( s_0 \).
(4) For a power-law potential \(U(s) = (s/s_0)^c\) with the unit length \(s_0\) and the power-law exponent \(c\), we find

\[
Q(s) = Ae^{-(s_0/s)^c - Bs}
\]

(35)

and

\[
S(\tau) = AVe^{-(s_0/(V\tau))^{c} - BV\tau}.
\]

(36)

Note that, in general, the normalization constant \(A\) and the Lagrange parameter \(B\) cannot be calculated analytically in this case [16].

Of course, in practical situation, one may find other potentials as well. Fig. 5, however, shows that the above cases already include a large variety of common time-gap distributions, including monotonously falling ones, distributions with a maximum and distributions with \(Q(0) = 0\).

4. Comparison with mathematical queuing theory

In contrast to the classical treatment of queues in mathematics, the consideration of a spatial dimension and interactions is rather a physical approach. In mathematical queuing theory, the interdeparture time statistics results from the stochastic arrival and exit processes. Each element in the queue is only characterized by a number, which reflects its position in the queue. Inspired by traffic models, however, it has been pointed out in the past that the description of material flows in economic or production systems sometimes also requires to take a spatial dimension into account [19–21].

Our calculations above have derived the statistics resulting from interactions between queuing elements in space, but they have neglected arrival and exit processes. This was for the sake of an analytical derivation of the gap distribution, which could be obtained for the special case of a circular queue thanks to the periodic boundary conditions in such a system. But how can we transfer our results to open systems, where elements can leave a queue in the front and others join the queue at its end?

In classical queuing theory, a queue is given by a stochastic arrival process, an inert buffer, and a stochastic departure process. Between the elements waiting in the queue, no interactions or distances are defined. In our physical treatment of queues, however, elements in the buffer do not only have a position in the queue, but also a distance from the service station at the downstream end of the queue, which defines distances between the elements as well. The elements are still entering the system with a given arrival rate \(\lambda\) and interarrival time distribution, but they are interacting in the buffer in dependence of the distance to their respective predecessor in the queue. Instead of specifying the departure rate \(\mu\) and the interdeparture time distribution, we specify the
speed $V$ at which the elements leave the system, i.e. we modify the treatment of the downstream boundary. Then, the interdeparture time distribution (the stochastic exit process) becomes a result of the gap distribution between the elements in the queue.\(^1\)

We can distinguish three regimes:

1. When there is only one or no element in the queue, the interdeparture time distribution is given by the interarrival time distribution, as each element moves at the constant speed $V$ along the physical extension of the buffer without any interactions, so that the distribution does not change.

2. In contrast, when the number of elements in the queue is large, we expect that the gap distribution in the bulk of the queue is independent of the arrival rate $\lambda$ and determined by the interactions among its elements according to the calculations above. However, the corresponding density $\rho$ of elements in the open queuing system is not anymore given by the number of elements, divided by the circumference of the circular system. It must be rather obtained as the inverse of the equilibrium distance resulting from the balance between the driving term $\left(\frac{v_0 - v_1}{\tau}\right)$ and the interaction forces $\gamma f(s_{i-1}) - f(s_i)$. For the velocity of the first element in the queue, we must insert the velocity $V$ imposed by the downstream boundary condition of the exit process. While in our previous calculations, the average velocity $V$ was a function of the density $\rho$, it is now the other way round. It should be also noted that the value of this velocity significantly influences (“controls”) the average departure rate $\mu \approx \rho(V)V$. Moreover, due to the stochastic arrival and exit processes, the number of the elements in the queue fluctuates in time, as for classical queues.

3. When the number of the elements in the queue is small but greater than 1, both the interarrival time distribution and the interactions among elements in the queue are expected to determine the interdeparture time distribution. For an example of an arrival-determined gap distribution, which is modified by hard-core interactions, see Ref. [22, Sec. IIB].

Obviously, the third case is the most interesting, suggesting a new research area for queuing theory in the future.

5. Summary and conclusions

We have presented a stationary solution for the gap distribution in long queues with interactions between its elements. This yields characteristic interdeparture time distributions, which may therefore be interpreted as effects of direct or indirect particle interactions. Our approach is applicable, if a solution with stationary distances and velocities is stable with respect to small perturbations, which is the case for small enough values of $\tau$ [15,23], but generalizations are necessary, if the velocity variance significantly depends on the density. Apart from a Gaussian shape of the velocity distribution, we have derived a formula for the distance distribution as a function of the interaction potential. This has been used to determine the interaction potential between vehicles on a freeway. Moreover, we have derived the associated time-gap distribution.

Our approach also opens perspectives for the optimization of queuing systems by modifying the interaction potential. In vehicle traffic, for example, the interactions between vehicles are changed by the application of driver assistance systems [24]. This allows one to reach a higher stability and capacity of traffic by influencing the time-gap distribution of vehicles, namely by reducing and homogenizing time gaps in order to reduce perturbations [25]. Similar methods may in the future be applied to increase the performance of production systems. Another interesting future research direction is to couple queuing systems with interactions to networks of queues as they are typically found in production systems. In such complex queuing networks, the interdeparture time distributions determine the interarrival time distributions of subsequent queues. Finally, it will be challenging to study queuing systems, which are not dominated by the interactions in the queue or by the interarrival time distribution, but where both aspects have an impact on the resulting interdeparture time statistics.

\(^1\)Alternatively, it is also possible to specify the stochastic exit process and to determine the resulting average velocity $V$ of departing elements, but this is more difficult to handle mathematically.
References