

Noisy Covariance Matrices and Portfolio Optimization

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Abstract. According to recent findings [1,2], empirical covariance matrices deduced from financial return series contain such a high amount of noise that, apart from a few large eigenvalues and the corresponding eigenvectors, their structure can essentially be regarded as random. In [1], e.g., it is reported that about 94% of the spectrum of these matrices can be fitted by that of a random matrix drawn from an appropriately chosen ensemble. In view of the fundamental role of covariance matrices in the theory of portfolio optimization as well as in industry-wide risk management practices, we analyze the possible implications of this effect. Simulation experiments with matrices having a structure such as described in [1,2] lead us to the conclusion that in the context of the classical portfolio problem (minimizing the portfolio variance under linear constraints) noise has relatively little effect. To leading order the solutions are determined by the stable, large eigenvalues, and the displacement of the solution (measured in variance) due to noise is rather small: depending on the size of the portfolio and on the length of the time series, it is of the order of 5 to 15%. The picture is completely different, however, if we attempt to minimize the variance under non-linear constraints, like those that arise e.g. in the problem of margin accounts or in international capital adequacy regulation. In these problems the presence of noise leads to a serious instability and a high degree of degeneracy of the solutions.

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1 Introduction

The concept of financial risk, which attempts to quantify the uncertainty of the outcome of an investment and hence the magnitude of possible losses, plays a fundamental role in finance today. Portfolio optimization aims at giving a recipe for the composition of portfolios such that the overall risk is minimized for a given reward, or, conversely, reward is maximized for a given risk. For example, the classical portfolio optimization problem formulated first by Markowitz [3] relies on the variance as a risk measure and expected return as a measure for reward. Since the return on a portfolio is a linear combination of the returns on the assets forming the portfolio with weights given by the proportion of wealth invested in the assets, the portfolio variance can be expressed as a quadratic form of these weights with the volatilities and correlations as coefficients. For any practical use of the theory, it will, therefore, be necessary to have reliable estimates for the volatilities and correlations, which, in most cases, are obtained from historical return series. Actually, volatility and correlation estimates extracted from historical data have become standard tools also for several other

risk management practices widely utilized in the financial industry.

Recently it has, however, been found by two independent groups [1,2] that empirical covariance matrices deduced from financial return series contain such a high amount of noise that, apart from a few large eigenvalues and the corresponding eigenvectors, their structure can essentially be regarded as random. In [1], e.g., it is reported that about 94% of the spectrum of correlation matrices determined from return series on the S&P 500 stocks can be fitted by that of a random matrix drawn from an appropriately chosen ensemble. In view of these striking results, the Markowitz portfolio optimization scheme based on a purely historical determination of the covariance matrix would seem to be totally inadequate [1,4], but the credibility of a number of standard risk management methodologies would also be shaken.

In this paper we will argue, however, that the impact of the results of [1,2] on the portfolio optimization problem may not be as dramatic as one might have expected. More specifically, it will be shown in a simulation example that for parameter values typically encountered in practice, the risk of the optimal portfolio determined in the presence of noise is usually no more than 5–15% higher than the risk without noise. Despite the high degree of noise of the covariance matrix, which translates indeed into a signifi-

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cant displacement in the weights of the optimal portfolio, the effect on the actual risk at the optimum is only of second order and therefore less pronounced. This suggests that some of the risk management methodologies based on empirical covariance matrices can actually be sufficiently accurate. The main purpose of this paper is to point out that the results of [1,2] and the practical usefulness of covariance matrices can, in fact, be reconciled.

2 Results and Discussion

We consider the following simplified version of the classical portfolio optimization problem: the portfolio variance $\sum_{i,j=1}^n w_i \sigma_{ij} w_j$ is to be minimized under the budget constraint $\sum_{i=1}^n w_i = 1$, where w_i denotes the weight of asset i in the portfolio while σ_{ij} represents the covariance matrix of returns (considered here as given). This means we exclude the riskless bond and seek the minimal risk portfolio in the space of risky assets. One might, of course, impose additional constraints (e.g. the usual one on the return), but the simplified form at hand provides the most convenient laboratory to test the effect of noise. The solution to the optimization problem can then be found using the method of Lagrange multipliers, and after some trivial algebra one obtains for the weights of the optimal portfolio:

$$w_i^* = \frac{\sum_{j=1}^n \sigma_{ij}^{-1}}{\sum_{j,k=1}^n \sigma_{jk}^{-1}}. \quad (1)$$

According to [1,2], correlation matrices determined from financial return series are such that apart from a few large eigenvalues and the corresponding eigenvectors, their structure is essentially random. Random matrix theory (RMT) allows one to calculate different eigenvalue and eigenvector statistics e.g. of a matrix $C_{ij} = \frac{1}{T} \sum_{t=1}^T x_{it} x_{jt}$ determined from series of random variables x_{it} independent and identically distributed, of mean zero and of unit variance ($i = 1, 2, \dots, n$ and $t = 1, 2, \dots, T$), see [1,2] and references therein. The observed deviations of empirical correlation matrices from RMT predictions [1,2,5,6] are due to genuine correlations between the financial series, while the apparently dominating random part can be interpreted as noise superimposed on these correlations. Therefore, any procedure using as input correlation matrices determined from financial return series will be biased by a significant amount of noise, and for the practical applicability of the procedure it would be highly desirable to know the magnitude of this bias.

In order to get an idea about the magnitude of the effect, we compare the solution obtained for a given noiseless covariance matrix $\sigma_{ij}^{(0)}$ with that obtained when noise is added (we call the new covariance matrix σ_{ij}). If, for example, $\sigma_{ij}^{(0)}$ is simply chosen to be an $n \times n$ identity matrix, noisy covariance matrices σ_{ij} can be generated as

$$\sigma_{ij} = \frac{1}{T} \sum_{t=1}^T x_{it} x_{jt}, \quad (2)$$

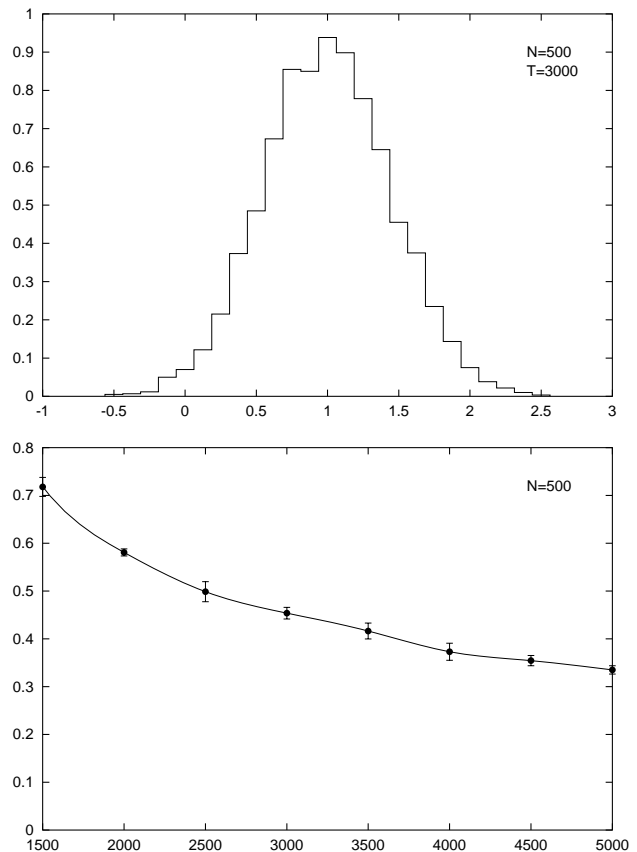


Fig. 1. (a) Histogram of nw_i^* illustrating the displacement of the solution in the presence of noise for $N = 500$ and $T = 3000$ (compare to $nw_i^{(0)*} = 1$). To achieve smoothness, the histogram has been averaged over 10 samples for σ_{ij} . (b) Standard deviation of nw_i^* for $N = 500$ and different values for T .

where $x_{it} \sim \text{i.i.d. } N(0, 1)$. (Of course, in the limit $T \rightarrow \infty$ the noise disappears and $\sigma_{ij} \rightarrow \sigma_{ij}^{(0)}$.) The solutions to the optimization problem in this setup are $w_i^{(0)*} = \frac{1}{n}$ and w_i^* given by Eq. (1), respectively. The difference between the two solutions shows the displacement of the optimal portfolio due to noise and provides a measure for the effect of noise on the optimization problem.

We have studied the behavior of nw_i^* for different system sizes N and different time series lengths T . Without noise this quantity is $nw_i^{(0)*} = 1$ for any i . In the presence of noise, however, nw_i^* oscillates around 1. The distribution of nw_i^* for typical values of N and T is given in Fig. 1(a). It can be seen from the figure that weights as low as 0 or as high as 2 appear with non-negligible frequency, which suggests that the effect of noise is quite strong, i.e. the optimal portfolio obtained using the noisy covariance matrix may be rather different from the “true” optimal portfolio. The standard deviation of this distribution as a function of T is given in Fig. 1(b). One can see that the deviation from the optimal portfolio remains significant even for quite large T .

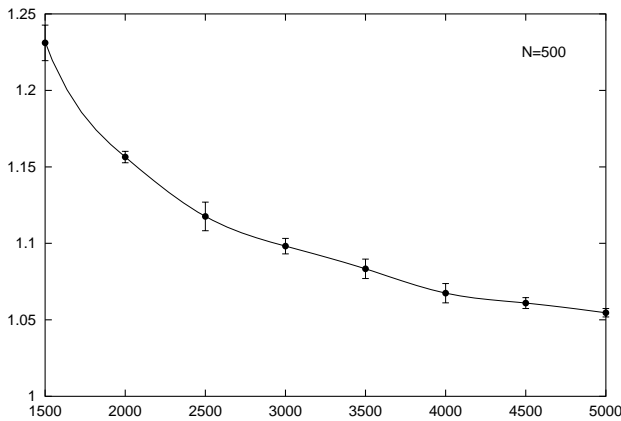


Fig. 2. Ratio $q = \frac{\sigma_p}{\sigma_p^{(0)}}$ quantifying the decrease in efficiency of the optimal portfolio due to noise in the covariance matrix as a function of T ($N = 500$).

However, the effect of noise should be assessed not so much on the basis of the change in composition, but rather of the shift in the volatility of the optimal portfolio, because this is the only factor rational investors should actually care about in our simplified optimization problem. Let us, therefore, compare the variance of the “true” (noiseless) optimal portfolio $\sigma_p^{(0)2} = \sum_{i,j=1}^n w_i^{(0)*} \sigma_{ij}^{(0)} w_j^{(0)*} = \sum_{i=1}^n w_i^{(0)*2}$ with the “true” variance of the optimal portfolio obtained in the presence of noise $\sigma_p^2 = \sum_{i,j=1}^n w_i^* \sigma_{ij}^{(0)} w_j^*$ $\sum_{i=1}^n w_i^{*2}$. More precisely, we have calculated the volatility ratio $q = \frac{\sigma_p}{\sigma_p^{(0)}}$ that measures the increase in volatility (and therefore decrease in efficiency) of the optimal portfolio due to noise. Fig. 2 shows the magnitude of this quantity as a function of T for N given. It can be seen from the figure that for $N = 500$ and $T > 3000$ the increase in volatility due to noise is less than 10%. The decrease in efficiency is, in most cases, of the order of 5–15% that seems reasonable from a practical point of view ($2000 < T < 5000$). It seems, therefore, that despite its obvious effect on the weights of the optimal portfolio, noise has a significantly less pronounced impact on the risk at the optimum.

It is interesting to estimate the magnitude of this effect for values of N and T similar to those considered in [1,2]. In [1] daily returns on $N = 406$ stocks of the S&P 500 over the period 1991–1996 (a total of $T = 1309$ daily observations) have been used for constructing the correlation matrix. We have found that for this portfolio size and for this time series length the value of q is around 1.20, i.e. the risk of the optimal portfolio in the presence of noise is about 20% larger than without. As for [2], in this paper 30-minute returns on the largest $N = 1000$ U.S. stocks over the two-year period 1994–1995 (a total of $T = 6448$ data points for each series) have been used, for which q is about 1.09, i.e. the decrease in efficiency due to noise is only around 9%. According to these findings, the impact of noise on the portfolio optimization problem does

not seem to be as dramatic as one might have feared and that, despite the high level of noise, empirical covariance matrices can still be used as input for a portfolio optimization problem without losing too much of accuracy.

Moreover, for smaller system sizes N we need shorter time series lengths T to achieve the same degree of precision. For example, for $N = 100$ and $T = 500$ the increase in volatility is 11%, while for $T = 1000$ it is only about 5%. Therefore, if one uses covariance matrices obtained e.g. from 4 years of daily returns (around 1000 observations) say on the S&P 100 stocks, the loss in efficiency due to noise in the covariance matrix is only about 5%.

In order to give a better representation of the actual structure of empirical correlation matrices, we have repeated our experiments with matrices which, in addition to the pure random part given by Eq. (1), have one clearly distinct eigenvalue chosen to be about 25 times larger than the largest eigenvalue predicted by RMT (see [1,2]), with a corresponding eigenvector in the direction of $(1, 1, \dots, 1)$. The displacement of the risk associated with the optimal portfolio due to noise has been found to be of the same order of magnitude as in the cases discussed earlier.

The explanation for the lack of a more dramatic effect on the portfolio in the presence of noise is actually very simple. A function $f(x)$ with a single well-defined flat minimum at x^* (e.g. a quadratic function) varies slowly in the neighbourhood of the minimum, therefore, the value $f(x)$ need not be much higher than $f(x^*)$, even for significant deviations of x from the minimum. In our case, the volatility of the portfolio is exactly such a function of the weights w_i , and therefore even if the weights deviate significantly from the optimal ones (as they do), the risk of the portfolio will not be dramatically affected. Since rational investors should not care about the composition of their portfolios but only about its risk, the effect of noisy covariance matrices on the portfolio optimization problem will be less significant than expected. In other words, it appears that despite noise, covariance matrices deduced from financial return series can have, in certain cases, a reasonable practical use.

Let us note, however, that the picture becomes completely different if we consider an optimization problem with non-linear constraints like those that arise e.g. in the case of margin accounts or in international capital adequacy regulation [7–9]. In these cases the budget constraint has the form $\sum_{i=1}^n \gamma_i |w_i| = 1$. As shown in [7], this problem maps exactly onto finding the ground states of a long-range spin glass, and now the presence of noise leads to a serious instability and a high degree of degeneracy of the solutions. The results of [7] have far reaching economic implications, the analysis of which are, however, far beyond the scope of this short note.

3 Conclusion

In this paper the impact of noisy covariance matrices on the portfolio optimization problem has been investigated. Earlier studies [1,2] have pointed out that a large part of the spectrum of empirical covariance matrices deduced

from financial return series corresponds to that of a purely random matrix. The presence of such a high level of noise in covariance data could have had devastating consequences for the reliability of different risk management practices based on the use of these matrices. We have analyzed the impact of this noise on the classical portfolio optimization problem and found that the risk of the optimal portfolio determined in the presence of noise is typically no more than 5–15% higher than in the absence of it, showing that the decrease in efficiency of the optimal portfolio is actually much less dramatic. This suggests that the important results of [1,2] and the practical usefulness of covariance matrices can, in fact, be reconciled.

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References

1. L. Laloux, P. Cizeau, J.-P. Bouchaud and M. Potters, *Phys. Rev. Lett.* **83**, 1467 (1999).
2. V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. N. Amaral and H. E. Stanley, *Phys. Rev. Lett.* **83**, 1471 (1999).
3. H. Markowitz, *Portfolio Selection: Efficient Diversification of Investments* (J. Wiley and Sons, New York, 1959).
4. L. Laloux, P. Cizeau, J.-P. Bouchaud and M. Potters, *Risk* **12**, No. 3, 69 (1999).
5. V. Plerou, P. Gopikrishnan, B. Rosenow, L. A. N. Amaral, T. Guhr and H. E. Stanley, e-print cond-mat/0108023.
6. S. Drożdż, F. Grümmer, F. Ruf and J. Speth, e-print cond-mat/0103606.
7. S. Galluccio, J.-P. Bouchaud and M. Potters, *Physica A* **259**, 449 (1998).
8. A. Gábor and I. Kondor, *Physica A* **274**, 222 (1999).
9. I. Kondor, *Int. J. of Theor. and Appl. Finance* **3**, 537 (2000).