Equilibrium Impact of Value-at-Risk Regulation

presented by
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joint work with
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Overview

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I. Introduction

A. Background


- Financial regulation aims at maintaining and improving the safety of the financial industry.

- Regulators advocate normative and simplified rules applicable to all financial institutions (banks).

- The Amendment recognizes the complexity of risk measurement and allows the use of internal risk models for reporting to regulatory authorities.

- Regulators ask for Value-at-Risk figures - the market standard - when internal models are used for market risk.
B. Motivation

• The common belief: Regulators do not take interactions into account. Banks are forced to sell when financial markets are experiencing disruptions. Feedback effects worsen the situation.

• Regulators assume risk management to be an exogenous problem, i.e. a game against nature.

• What happens in a general dynamic model when feedback effects and intertemporal behavior are taken into account?

~ What we need: A model illustrating the impact of market risk regulation on equilibrium quantities.
C. Existing Literature


Model:
- Dynamic portfolio problem.
- Optimization of CRRA-utility subject to VaR constraint.
- Complete markets with lognormal prices.

Results:
- VaR constraints results in fattening the left tail.
- Expected shortfall (ES) would remove this shortcoming.
- Risk measures for regulation should be based on ES.

Shortcomings:
- Lognormal stock prices.
- Static formulation of dynamic portfolio problem.
2. Cuoco, He, and Issaenko (2001)

**Model:**
- Dynamic portfolio problem with VaR updating.
- Optimization of CRRA-utility subject to VaR constraint.
- Complete markets with lognormal prices.

**Results:**
- VaR constraints do not distort incentives.
- No need for ES based regulation.

**Shortcomings:**
- Lognormal stock prices.
- No general equilibrium analysis as in Basak and Shapiro (2001).
- Numerical solutions.
3. Danielsson, Shin, and Zigrand (2001)

**Model:**
- Series of myopic VaR optimization with backward-looking belief revision.
- Optimization of CARA-utility subject to VaR constraint.
- Simulation of Equilibrium with GARCH price dynamics.

**Results:**
- VaR constraints exacerbate financial shocks.
- Uniformity of regulation leads to distortion of financial system.

**Shortcomings:**
- All results based on simulation of “multiperiod-myopic” investors.
- Numerical solutions.
- Not a full equilibrium model.
- No forward-looking anticipation of VaR constraints.
D. Our Contribution

• Partial equilibrium analysis with
  – dynamically adjusted VaR-limits.
  – model framework for stochastic volatility.
  – perturbative solutions to portfolio optimization problem with both utility from end-of-period wealth and intermediate “consumption”.

• General equilibrium of a continuous-time model with
  – exchange economy with intermediate consumption.
  – heterogeneous investors.
  – model framework for stochastic volatility.
  – VaR regulation becomes endogenous.

• Implications of regulation on stock price dynamics and interest rate in the presence of stochastic volatility.
II. Model Setup

Price Process and State-Variable

• Financial market consists of cash-bond $B$ and risky asset $P$ following diffusion processes.

• The dynamics of $B_t$ and $P_t$ are

\[
dB_t = r(X_t)B_t dt, \quad dP_t = \alpha(X_t)P_t dt + \sigma(X_t)P_t dZ_t, \quad (1)
\]

with state variable $X_t$,

\[
dx_t = \mu_X(X_t)dt + \sigma_X(X_t)dZ_t^X, \quad E[dZ_t dZ_t^X] = \rho dt. \quad (2)
\]

• The bank chooses a portfolio $(1 - w_t, w_t)$ invested in $(B_t, P_t)$. Value of portfolio equals bank’s wealth $W$. 
Regulatory Constraints

The standard tool for regulatory reporting is Value-at-Risk:

**Definition:**

$$\text{VaR}_{t}^{\nu,w}$$ is the maximum potential loss that a portfolio \((1 - w_t, w_t)\) can suffer in the \(100(1 - \nu)\%\) best cases in \(\tau \times 250\) days.

or:

$$\text{VaR}_{t}^{\nu,w}$$ is the minimum potential loss that a portfolio \(X\) can suffer in the \(100\nu\%\) worst cases in \(\tau \times 250\) days.

**Formal Definition:**

The time-\(t\) Value-at-Risk of a portfolio \(w_t\) for a given \(\mathbb{P}\)-probability level \(\nu \in (0, 1)\) and over a fixed time-horizon \(\tau > 0\) is defined by

$$\text{VaR}_{t}^{\nu,w} = \inf \{ L \geq 0 | \mathbb{P} (W_t - W_{t+\tau} \geq L | \mathcal{F}_t) < \nu \},$$

where \(W_{t+\tau}\) is the portfolio value at time \(t + \tau\) of a fixed-weight strategy with initial weight \(w_t\) at time \(t\).
A few Comments,

- The above definition is consistent with the VaR concept used in practice for regulatory purposes. Time horizon $\tau$ is chosen to be 1 or 10 days.
- We assume the bank to continuously calculate VaR over time.
- We define a risk-limit as $\beta = \frac{\text{VaR}_{t}(W, t)}{W_t} \in [0, 1)$.

...a first Problem,

- No closed-form solution for VaR are available under our general assumptions about price dynamics.

...and it’s “Solution” ...

- Express first-order VaR constraint as explicit lower and upper bound for portfolio fractions:
  \[ w^-_b(X, t) \leq w_t(X, t) \leq w^+_b(X, t). \]
- Expressions for error bounds available.
 Portfolio constraints

Figure 1: The VaR constraint in terms of portfolio fractions. The figure plots the portfolio fractions $w_b^+(X,t)$, when we assume $r(X_t) = r$, $\lambda(X_t) = \lambda X_t^2$, and $\sigma(X_t) = \sigma X_t$.
III. Partial Equilibrium

A. The Bank’s Optimization Problem

- The bank derives utility from final wealth $W_T$ and current expenditures:

$$V(W, C) = \int_0^T e^{-\delta s} \frac{C_s^\gamma - 1}{\gamma} ds + e^{-\delta T} \frac{W_T^\gamma - 1}{\gamma}, \quad 0 \leq \delta < 1,$$

(3)

where $\delta$ is the constant subjective discount rate, and $C_t$ is the flow of consumption.

- The control problem considered:

\[
\begin{align*}
(P1) : \quad & J(W, X, t) = \max_{w, c} \mathbb{E} \left[ V(W, C) | \mathcal{F}_t \right], \\
& \text{s.t.} \quad w_b^- \leq w_t \leq w_b^+, \ t \in [0, T], \\
& \quad \quad dW_t = (w_t \lambda(X_t) W_t + (r(X_t) - c(X_t) W_t) dt + w_t \sigma(X_t) W_t dZ_t, \\
& \quad \quad dX_t = \mu_X(X_t) dt + \sigma_X(X_t) dZ_t^X.
\end{align*}
\]
...other problems arise:

a) find the explicit solution for $J$ and hence $w_f(X, t)$.

b) show that $J$ is increasing and strictly concave.

ad a) ...our approach: Perturbation Theory

✓ We perturb the solution of the log-investor (known explicitly) w.r.t. the risk-aversion parameter $\gamma$.

✓ Solutions are available in quasi-closed form up to arbitrary order.

✓ This allows us to study higher -order approximation, their accuracy, and their convergence.

✓ Comparative statics are exact!

ad b) ... ✓
Figure 2: Comparing perturbation approach with Monte Carlo simulation. We assume \( \alpha(X_t) = 0.05X_t^2, \sigma(X_t) = 0.25X_t, \theta = \kappa = 0.8, \sigma_X = 0.2, T = 1. \)
Perturbative Solution

Proposition 1. The first-order approximations of the optimal policies are given by

\[ w^{(1)}(X,t) = (1 + \gamma)X^{n_1 - 2n_2} \frac{\lambda}{\sigma^2} + \gamma X^{m - n_2} \frac{\rho \sigma X}{\sigma} \frac{\partial g_0(X,t)}{\partial X}, \]
\[ c^{(1)}(X,t) = \frac{1 - \gamma (g_0(X,t) + \log A_t)}{A_t}, \]

where \( A_t = e^{-\delta(T-t)} + \frac{1 - e^{-\delta(T-t)}}{\delta} \), \( g_0(X,t) = g^f_0(X,t) + g^c_0(X,t) \), and

\[ g^f_0(X,t) = \frac{1}{A_t} e^{-\delta(T-t)} \mathbb{E} \left[ H^f_{t,T} | \mathcal{F}_t \right] + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} \left( \mathbb{E} \left[ H^f_{t,s} | \mathcal{F}_t \right] - \log A_s \right) ds, \]
\[ g^c_0(X,t) = \frac{1}{A_t} e^{-\delta(T-t)} \mathbb{E} \left[ H^c_{t,T} | \mathcal{F}_t \right] + \frac{1}{A_t} \int_t^T e^{-\delta(s-t)} \left( \mathbb{E} \left[ H^c_{t,s} | \mathcal{F}_t \right] - \log A_s \right) ds, \]

\[ \mathbb{E} \left[ H^f_{t,T} | \mathcal{F}_t \right] = r(T - t) - \int_t^T A_s^{-1} ds + \frac{1}{2} \lambda^2 \int_t^T \mathbb{E} \left[ X_s^{2(n_1 - n_2)} | \mathcal{F}_t \right] ds, \]
\[ \mathbb{E} \left[ H^c_{t,T} | \mathcal{F}_t \right] = -\frac{1}{2} \int_t^T \mathbb{E} \left[ \mathbb{1}_{\{\phi(X_s,s) < 0\}} \phi(X_s,s)^2 | \mathcal{F}_t \right] ds \]

\[ \phi(X,t) = \frac{v}{\sqrt{\tau}} + \frac{\sqrt{(\lambda X^{n_1} \tau + \sigma X^{n_2} \sqrt{\tau} v)^2 + 2\sigma^2 X^{2n_2} \tau ((r - A_t^{-1}) \tau - \log(1 - \beta))}}{\sigma X^{n_2} \tau}. \]
B. Do VaR Constraints Distort Incentives?

Asymptotic partial equilibrium problem reveals: Before hitting the VaR constraints:

i) a bank with $\gamma > 0$ ($\gamma < 0$) increases (decreases) optimal consumption $c_t$.

ii) the bank’s optimal portfolio is unaffected if $n_2 = n_1$.

iii) the bank’s optimal portfolio is unaffected if $\gamma = 0$.

iv) given $n_1 \neq n_2$, the bank increases or decreases its exposure to the risky asset according to:

| $\frac{\partial g_0}{\partial X}$ | $\frac{\partial g_0}{\partial X}$ | $\gamma \rho > 0$ | $\gamma \rho < 0$
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Figure 3: Portfolio policy under VaR and the influence of investment horizon. Graphics on the left assume a positive correlation of $\rho = 0.4$. Graphics on the right assume $\rho = -0.4$. (A) and (B) assume an investment horizon of $T = 1$ year. The next two graphics assume $T = 5$, and the final two graphics assume $T = 10$. The solid bold line represents the portfolio policy of the VaR constrained bank. At the circled point, the bank runs into the VaR constrained represented by the dotted line. The dashed line represents the optimal policy in case there were no constraints at all. The parameters were chosen as follows: $\nu = 0.01$, $\beta = 0.05$, $\tau = 10/250$, $r = 0.05$, $\lambda = 0.03$, $\sigma = 0.32$, $\kappa_X = \theta_X = 0.2$, $\sigma_X = 0.6$. The calculations were performed using standard Monte Carlo methods.
Figure 4: Portfolio policy under VaR and the influence of confidence level. Graphics on the left assume $\rho = 0.4$ and $\rho = -0.4$ on the right side. (A) and (B) assume a confidence bound of $\nu = 0.01$. (C) and (D) assume $\nu = 0.05$. (E) and (D) show the differences in optimal policies when the confidence level is tightened, i.e. we plot the differences (C)-(A) and (D)-(B). We assumed the same parameters as in Figure 3. The time horizon is set to $T = 5$ years.
Results in Partial Equilibrium

• We remove the time-inconsistency of Basak and Shapiro (2001) and extend the constant volatility model of Cuoco, He, and Issaenko (2001).

• We provide asymptotic solutions and consider their accuracy.

• The presence of VaR based regulation can have different effects on the behavior of banks. The sign of the effects depend on the risk aversion coefficient and the correlation between volatility and asset price changes.

• We found no evidence within our model that the distortions in the bank’s policies would be removed by using Expected Shortfall as risk measure.

• But we have to be careful:

  ► We are still in a partial equilibrium! Drift, interest rate and volatilities are the outcome of interplay between market participants!

  ► VaR is an explicit function of quantities that are determined in equilibrium!
General Equilibrium

A. Model Construction

Financial and Good Markets

Optimizing Banks ($I$, $II$)

Market Clearing Conditions

\[
\begin{align*}
    c_I^t W_I^t + c_{II}^t W_{II}^t &= e_t + O(\gamma^2) \\
    w_I^t \omega_I^t + w_{II}^t \omega_{II}^t &= 1 + O(\gamma^2)
\end{align*}
\]

State Variables

\[
\omega^I, X
\]

Market Prices

\[
\frac{dP_t}{P_t} + e_t dt = \alpha(\chi, t) dt + \sigma(\chi, t)^\top dZ_t
\]
B. Do VaR Constraints Distort the Economy?

Regulation can only be successful if considered along the following dimensions:

- Regulatory Control: $\beta, \nu, \tau$
- Market Participants: $\gamma, \omega$
- Market Factors: $X, \rho X$
1. Pure Log-Economy

- Unregulated institution increases its exposure.
- Interest rates fall.
- Shift in cross-sectional wealth regardless of risk-aversion:
  - wealth is shifted to the unregulated institution.
  - volatility of wealth dynamics increases!
- Risk premium increases, as drift and volatility of stock price process remain unaffected.

2. Positive Correlation $\rho_{eX}$

+ Increase of stock price’s drift rate.
+ Increase of stock price’s volatility.

3. Negative Correlation $\rho_{eX}$

+ Decrease of stock price’s drift rate.
+ Decrease of stock price’s volatility.
Figure 5: Impact of VaR regulation in a log-economy $M_0$. We assumed the following parameter values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.05, T = 5, \tau = 1/250, \beta = 2.5\%, v = 5\%, \omega = 0.5, \gamma_I = \gamma_{II} = 0$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, drift and volatility of the cross-sectional wealth dynamics, the dotted lines to the right of point $A$ represent the corresponding quantities in the unconstrained economy.
Figure 6: Impact of VaR regulation for $M_1$ with positive correlation. We assumed $d e_t / e_t = \mu_e dt + \sigma_e X_t dZ_t^e$ and the following values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.1, T = 2, \tau = 1/250, \beta = 5\%, v = 1\%, \omega = 0.5, \rho_{eX} = 0.4, \gamma_I = -0.1, \gamma_{II} = 0.4$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, drift and volatility of the asset price process, the dotted lines to the right of point A represent the corresponding quantities in the unconstrained economy.
Figure 7: Impact of VaR regulation on Sharpe Ratio for $M_1$ with positive correlation. We assumed the same parameter values as for Figure 6. The bold line is the Sharpe Ratio resulting from the constrained economy. The dotted line to the right of point $A$ represents the Sharpe Ratio in the unconstrained economy.
Figure 8: Impact of VaR regulation for $M_2$ with negative correlation. We assumed $de_t/e_t = \mu_e X_t dt + \sigma_e X_t dZ_t^e$ and the following values: $\sigma_e = 0.2, \mu_e = 0.1, \sigma_X = 1, \theta = \kappa = 0.1, \delta = 0.1, T = 2, \tau = 1/250, \beta = 2.5\%, v = 1\%, \omega = 0.5, \rho_{eX} = -0.4, \gamma_I = -0.1, \gamma_{II} = 0.4$. Bold lines represent quantities resulting from the constrained economy. In the panels for the interest rate, drift and volatility of the asset price process, the dotted lines to the right of point $A$ represent the corresponding quantities in the unconstrained economy.
Summary

- We consistently formulate the bank’s optimization problem in continuous-time framework.
- We provide asymptotic analysis of partial and general equilibrium models with VaR regulation and stochastic opportunity set.
- The conclusions on the impact of VaR regulation on the economy are mixed, rationalizing some puzzles.
- To guarantee effectiveness of VaR regulation, a careful analysis of market participants and market situation is required.
- Certainly, the current (ad-hoc) regulatory framework does not meet these requirements!
References

